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One-Dimensional XY Model: Ergodic Properties and Hydrodynamic Limit

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We prove theorems on convergence to a stationary state in the course of time for the one-dimensional XY model and its generalizations. The key point is the well-known Jordan–Wigner transformation, which maps the XY dynamics onto a group of Bogoliubov transformations on the CAR C^*-algebra over Z^1. The role of stationary states for Bogoliubov transformations is played by quasifree states and for the XY model by their inverse images with respect to the Jordan–Wigner transformation. The hydrodynamic limit for the one-dimensional XY model is also considered. By using the Jordan–Wigner transformation one reduces the problem to that of constructing the hydrodynamic limit for the group of Bogoliubov transformations. As a result, we obtain an independent motion of "normal modes," which is described by a hyperbolic linear differential equation of second order. For the XX model this equation reduces to a first-order transfer equation.

KEY WORDS: Nonequilibrium quantum statistical mechanics; convergence to a stationary state; hydrodynamic limit; one-dimensional XY model.

1. INTRODUCTION

The one-dimensional XY model has attracted the attention of many authors.[1-11] The interest in this model is based on its connection to the model of "quasifree" motion, which is established via the Jordan–Wigner transformation (see Ref. 12, Chapter 6.1, Section 6.2.1 and Notes and Remarks on Chapter 6.1). In particular, this connection allows one to investigate certain dynamic properties of the XY model: the convergence to

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a stationary state and the hydrodynamic approximation. These problems are the theme of this paper.

The problem of convergence to a stationary state can be stated for systems with infinitely many degrees of freedom as follows. First, one has to construct a group of *-automorphisms $\{S_t, t \in R^1\}$ of the corresponding C^*-algebra (in our case the quasilocal spin C^*-algebra $\mathfrak{M}$, CAR C^*-algebra $\mathfrak{U}^-$, or CCR C^*-algebre $\mathfrak{U}^+$), which is generated by the (formal) Hamiltonian of the model (more precisely, by the corresponding derivation on the C^*-algebra). Constructing such a group is a nontrivial problem, which is solved at present in an appropriate sense only for some classes of quantum systems. If an initial state of the system is given, then we want to study the time-evolved state S_t^*Q as $t \to \pm\infty$. A physically natural conjecture is that in a "generic" situation the state S_t^*Q converges to an equilibrium Gibbs state (or to a mixture of such states).

To prove the convergence as $t \to \pm\infty$ seems to be a difficult problem, which can be solved only for a few special classes of models. As a first one, we mention the free motion model[13–15] and the class of linear models (groups of Bogoliubov transformations[16,17]), which includes the free gas and harmonic oscillators. Among nonlinear models the simplest ones are the one-dimensional XY model[1–11] and the model of one-dimensional (quantum) hard rods[18]. A survey of related results is given in Ref. 19.

A separate subject is the study of "local perturbations," which has been initiated in Ref. 20.

An interesting feature of the above models is that the set of stationary states is not exhausted by the equilibrium Gibbs states. For instance, the set of stationary states for a (nondegenerate) group of Bogoliubov transformations contains a large family of quasifree invariant states.[16,17] For the XY model the stationary states are obtained from quasifree states by means of the Jordan–Wigner transformation. A similar construction can be carried through for hard rods.[18]

In Section 2 we prove a theorem on convergence to a stationary state for the one-dimensional XY model and its generalizations. This is a general result on convergence from the point of view of conditions on the Hamiltonian (i.e., on the group $\{S_t\}$) and on the initial state Q. Such a result is based on general convergence theorems proven for groups of Bogoliubov transformations.[17]

Section 3 is devoted to the problem of a hydrodynamic description of time evolution in the XY model and its generalizations. For the history of the formulation of this problem we refer the reader to Refs. 21–23 and to the review in Ref. 24. Notice that in the XY model one gets a somewhat unusual hydrodynamic equation: this is explained by the degenerate character of this model, namely, by the abundance of stationary states (see above). For the isotropic XX model we obtain an equation for the transfer of "normal modes" along straight lines on the cylinder $R^1 \times [-\pi, \pi)$. For the general XY model a more general differential equation results, which is linear, of second order, and of hyperbolic type, and describes an independent motion of normal modes.

It is worth noting that the hydrodynamic equation we get for the XY model is similar to the equation obtained for the classical linear models (see Ref. 22). We expect that the same is true for one-dimensional hard rods: the quantum hydrodynamic equation will have a similar form to the classical one (see Ref. 21). This leads to the conjecture that the quantum effects are, in a sense, negligible in the hydrodynamic regime (at least for degenerate models). However, one should be very careful at this point: the coincidence of classical and quantum hydrodynamics may be the consequence of the fact that the equilibrium states are described by similar parameters. If this "rule" is violated (i.e., if one considers a quantum model with no direct classical analog), then the quantum hydrodynamics might be different from the classical one.

In Section 4 we briefly discuss some examples of initial states (and families of initial states) for which the conditions in the theorems on the convergence to a stationary state and on the hydrodynamic approximation are fulfilled.

2. PRELIMINARIES. CONVERGENCE TO STATIONARY STATES FOR ONE-DIMENSIONAL *XY* MODEL AND ITS GENERALIZATIONS

Let $\mathscr{M}$ denotes the complex 2×2 matrix algebra. The C^*-algebra of the one-dimensional quantum spin-1/2 system is defined as the infinite tensor product

$$\mathfrak{M} = \mathscr{M}^{\otimes Z^1} \tag{2.1}$$

The local $*$-subalgebra of $\mathfrak{M}$ is denoted by $\mathfrak{M}^0$ and the C^*-subalgebra corresponding to a "volume" $I \subset Z^1$ by $\mathfrak{M}_I$. As usual, we denote by σ_j^x, σ_j^y, and σ_j^z the Pauli matrices associated with the site $j \in Z^1$. Consider the derivation δ on $\mathfrak{M}$ given by

$$\delta A = i[H, A] \tag{2.2}$$

where H is the (formal) Hamiltonian of the one-dimensional XY model with an extra magnetic field,

$$H = \sum_{j \in Z^1} (\alpha \sigma_j^x \sigma_{j+1}^x + \beta \sigma_j^y \sigma_{j+1}^y + h_0 \sigma_j^z), \qquad \alpha, \beta, h_0 \in R^1 \tag{2.3}$$

Although the series (2.3) diverges, the formula (2.2) is correct due to locality of A. The one-parameter group of *-automorphisms $\{W_t, t \in R^1\}$ of $\mathfrak{M}$ generated by δ determines the dynamics of the one-dimensional XY model. The time evolution of a given state Q on $\mathfrak{M}$ is defined by

$$W_t^* Q(A) = Q(W_{-t} A), \qquad A \in \mathfrak{M} \tag{2.4}$$

Denote by ϕ the *-automorphism on $\mathfrak{M}$ defined as

$$\phi(A) = \prod_{j=-\infty}^{\infty} \sigma_j^z A \prod_{j=-\infty}^{\infty} \sigma_j^z \tag{2.5}$$

As above, this definition is correct due to locality of A. Let $\mathfrak{M}(+)$ be the C^*-subalgebra of $\mathfrak{M}$ consisting of ϕ-invariant elements. It is easy to verify that $\mathfrak{M}(+)$ is W_t-invariant.

In the course of the analysis of the one-dimensional XY model a key role is played by the so-called Jordan–Wigner transformation, which induces a *-isomorphism between $\mathfrak{M}(+)$ and the even C^*-subalgebra $\mathfrak{U}_{\mathrm{ev}}^-$ of the CAR C^*-algebra $\mathfrak{U}^-$ over Z^1. The C^*-algebra $\mathfrak{U}^-$ is defined in the following way. Let U denote the Hilbert space $l_2(Z^1)$ and $\mathscr{H}_- = \exp_-^{\oplus} U$ the fermion Fock space over U. In $\mathscr{H}_-$ one defines the action of fermion creation and annihilation operators $a^+(h)$, $a(h)$, $h \in U$, which satisfy the CAR [$a^+(h)$ depends on h linearly, and $a(h)$ antilinearly). The C^*-algebra $\mathfrak{U}^-$ is generated by $\{a^+(h), a(h)\}$, or, equilivalently, by the operators $a_j^+ = a^+(e_j)$, $a_j = a(e_j)$, $j \in Z^1$, where $\{e_j\}$ is the "canonical" basis in U. The local *-subalgebra of $\mathfrak{U}^-$ is denoted by $\mathfrak{U}^{-0}$. The even C^*-subalgebra $\mathfrak{U}_{\mathrm{ev}}^-$ of $\mathfrak{U}^-$ is defined as that generated by monomials $a_j^\# a_k^\#$, where $a_l^\#$ means (independently) a_l^+ or a_l. The Jordan–Wigner transformation is written formally as

$$a_j^+ \leftrightarrow \prod_{s<j} \sigma_s^z \sigma_j^+, \qquad a_j \leftrightarrow \prod_{s<j} \sigma_s^z \sigma_j^- \tag{2.6}$$

where

$$\sigma_j^+ = \frac{1}{2}(\sigma_j^x + i\sigma_j^y), \qquad \sigma_j^- = \frac{1}{2}(\sigma_j^x - i\sigma_j^y)$$

For second-order monomials $a_j^\# a_k^\#$ this transformation is correct. In fact, it suffices to write down the corresponding formulas for the monomials $a_j^+ a_k^+$ and $a_j^+ a_k$ with $j \leqslant k$; for other cases one may use CAR and conjugation:

$$a_j^+ a_k^+ \leftrightarrow \sigma_j^+ \sigma_{(j-1,k)}^z \sigma_k^+, \qquad a_j^+ a_k \leftrightarrow \sigma_j^+ \sigma_{(j-1,k)}^z \sigma_k^- \tag{2.7}$$

Here (and below) $\sigma^z_{(j,j')}$, $j<j'-1$, denotes the product $\prod_{j<n<j'} \sigma^z_n$; for $j'-j=1$ we set $\sigma^z_{(j,j')}=\mathbb{1}$ (the identity 2×2 matrix).

It is possible to verify (see, e.g., Ref. 12, Example 6.2.14) that (2.7) defines a *-isomorphism between the C^*-algebras $\mathfrak{M}(+)$ and $\mathfrak{A}^-_{\text{ev}}$. This map is denoted by ψ and will play a crucial role in what follows.

On both C^*-algebras $\mathfrak{M}$ and $\mathfrak{A}^-$ the standard action of the space translation group is defined. In both cases we denote this group by $\{U_j, j\in Z^1\}$. The C^*-algebras $\mathfrak{M}(+)$ and $\mathfrak{A}^-_{\text{ev}}$ are U_j-invariant and the *-isomorphism $\psi: \mathfrak{M}(+)\to \mathfrak{A}^-_{\text{ev}}$ commutes with $\{U_j\}$.

In general, by using the *-automorphism ψ, one obtains a one-to-one correspondence between the groups of *-automorphisms of $\mathfrak{M}(+)$ and $\mathfrak{A}^-_{\text{ev}}$. Furthermore, one obtains a one-to-one-correspondence between the states of these C^*-algebras, or, which is more convenient, between ϕ-invariant states of $\mathfrak{M}$ and even states of $\mathfrak{A}^-$.

The isomorphism ψ maps the automorphisms $W_t: \mathfrak{M}(+)\to \mathfrak{M}(+)$ onto Bogoliubov transformations (canonical linear transformations) of the C^*-algebra $\mathfrak{A}^-_{\text{ev}}$. More precisely, the group of *-automorphisms $\{W_t\}$ of $\mathfrak{M}(+)$ generated by the derivation (2.2) is transformed into the group of *-automorphisms $\{\mathscr{T}_t\}$ of $\mathfrak{A}^-_{\text{ev}}$ generated by the derivation

$$\gamma A = i[G, A], \qquad A\in \mathfrak{A}^{-0}\cap \mathfrak{A}^-_{\text{ev}} \tag{2.8}$$

where G is the (formal) quadratic Hamiltonian

$$\begin{aligned} G = &\sum_{j,k\in Z^1: j<k} [g^{(1)}(k-j)\, a_j^+ a_k^+ + g^{(1)}(k-j)^-\, a_k a_j] \\ &+ \sum_{j,k\in Z^1} g^{(2)}(k-j)\, a_j^+ a_k \end{aligned} \tag{2.9}$$

The functions $g^{(1)}, g^{(2)}: Z^1\to R^1$ are given by

$$\begin{aligned} g^{(1)}(j) &= \pm(\beta-\alpha), \qquad & j&=\pm 1 \\ &=0, & j&\neq \pm 1 \end{aligned} \tag{2.10a}$$

$$\begin{aligned} g^{(2)}(j) &= -(\alpha+\beta), \qquad & j&=\pm 1 \\ &=2h_0, & j&=0 \\ &=0, & j&\neq 0, \pm 1 \end{aligned} \tag{2.10b}$$

The group of *-automorphisms $\{\mathscr{T}_t\}$ generated by the derivation (2.8) transforms the elements a_j^+, a_k, j, $k\in Z^1$, according to a linear law (precisely this property distinguishes the Bogoliubov transformations among general *-automorphisms of $\mathfrak{U}^-$):

$$\mathscr{T}_t a_j^+ = a^+(T_t^{(1)} e_j) + a(T_t^{(2)} e_j) \tag{2.11a}$$

$$\mathscr{T}_t a_k = a^+(T_t^{(2)} e_k) + a(T_t^{(1)} e_k) \tag{2.11b}$$

where $T_t^{(1)}$ is a linear and $T_t^{(2)}$ an antilinear bounded operator on U.

Equations (2.11a) and (2.11b) allow us to write a more convenient representation for the derivation (2.8). Consider the operator 2×2 matrix

$$\mathbb{T}_t = \begin{pmatrix} T_t^{(1)} & T_t^{(2)} \\ T_t^{(2)} & T_t^{(1)} \end{pmatrix} \tag{2.12}$$

The matrix family $\{\mathbb{T}_t, t \in R^1\}$ forms the one-parameter group and hence may be written as[3] $\mathbb{T}_t = \exp(it\mathbb{D})$, where the matrix $\mathbb{D}$ reads

$$\mathbb{D} = \begin{pmatrix} B & C \\ C & B \end{pmatrix} \tag{2.13}$$

The operators B and C are given in the Fourier representation by

$$\hat{B}f(\theta) = 2[h_0 - (\alpha + \beta)\cos\theta]\, f(\theta), \qquad \theta \in [-\pi, \pi) \tag{2.14a}$$

$$\hat{C}f(\theta) = 2i(\alpha - \beta)\sin\theta\, f(-\theta)^-, \qquad \theta \in [-\pi, \pi) \tag{2.14b}$$

For the case $\alpha = \beta$ (XX model) the infinitesimal matrix $\mathbb{D}$ becomes diagonal. The corresponding group of the Bogoliubov transformations may be regarded as an immediate analog of the so-called free motion.

Equations (2.8)–(2.9) lead to a natural generalization of the XY model. Consider the hamiltonian $\tilde{G}$ of the form (2.9) where $\hat{g}^{(1)}$ is an odd and $\hat{g}^{(2)}$ a real function on $[-\pi, \pi)$, and are assumed to be smooth enough. The corresponding derivation $\tilde{\gamma}$ [see 2.8] generates the group of Bogoliubov transformations $\tilde{\mathscr{T}}_t$: $\mathfrak{U}_{\mathrm{ev}}^- \to \mathfrak{U}_{\mathrm{ev}}^-$, $t \in R^1$. As above, this group is determined by a group of operator matrices $\{\mathbb{T}_t\}$ of the form (2.12). The generator $\tilde{\mathbb{D}}$ in the general case is of the form (2.13), where $\tilde{B}$ and $\tilde{C}$ are given in the Fourier representation by

$$\hat{\tilde{B}}f(\theta) = \hat{g}^{(2)}(\theta)\, f(\theta), \qquad \theta \in [-\pi, \pi) \tag{2.15a}$$

$$\hat{\tilde{C}}f(\theta) = -\hat{g}^{(1)}(\theta)\, f(-\theta)^-, \qquad \theta \in [-\pi, \pi) \tag{2.15b}$$

[3] The formal proof of this is to pass to a corresponding group of linear operator 2×2 matrices (e.g., by multiplying from both sides by diagonal idempotent matrices) and then to use the Stone theorem.

which generalizes (2.14a) and (2.14b). By using the isomorphism ψ we obtain the corresponding group of *-automorphisms $\tilde{W}_t: \mathfrak{M}(+) \to \mathfrak{M}(+)$, $t \in R^1$. It is not hard to check that the infinitesimal derivation $\tilde{\delta}$ for $\{\tilde{W}_t\}$ is determined by the Hamiltonian $\tilde{H}$ of the form

$$\begin{aligned}\tilde{H} = \sum_{j,k \in Z^1: j<k} & [f^{(1)}(k-j)\, \sigma_j^x \sigma_{(j,k)}^z \sigma_k^x \\ & + f^{(2)}(k-j)\, \sigma_j^x \sigma_{(j,k)}^z \sigma_k^y + f^{(3)}(k-j)\, \sigma_j^y \sigma_{(j,k)}^z \sigma_k^x \\ & + f^{(4)}(k-j)\, \sigma_j^y \sigma_{(j,k)}^z \sigma_k^y] + h_0 \sum_{j \in Z^1} \sigma_j^z \end{aligned} \tag{2.16}$$

where $h_0 \in R^1$ and the functions $f^{(s)}: Z_+^1 \to R^1$, $s = 1,..., 4$, are given by

$$h_0 = \frac{1}{2} g^{(2)}(0) \tag{2.17a}$$

$$f^{(1)} = -\frac{1}{2} \operatorname{Re}(g^{(1)} + g^{(2)}), \qquad f^{(2)} = \frac{1}{2} \operatorname{Im}(g^{(1)} - g^{(2)}) \tag{2.17b}$$

$$f^{(3)} = \frac{1}{2} \operatorname{Im}(g^{(1)} + g^{(2)}), \qquad f^{(4)} = \frac{1}{2} \operatorname{Re}(g^{(1)} - g^{(2)}) \tag{2.17c}$$

The Hamiltonian $\tilde{H}$ of the form (2.16) and the group $\{\tilde{W}_t\}$ are called the Hamiltonian and dynamics of the generalized XY model, respectively. In the sequel we shall need some nondegeneracy conditions, which will be formulated in terms of functions

$$\omega_\pm = \operatorname{Od} \hat{g}^{(2)} \pm w \tag{2.18a}$$

where

$$w = [(\operatorname{Ev} \hat{g}^{(2)})^2 + |\hat{g}^{(1)}|^2]^{1/2} \tag{2.18b}$$

and Ev $\hat{g}$ and Od $\hat{g}$ denote, respectively, the even and odd parts of $\hat{g}$. Namely, we suppose that the following condition (A) is fulfilled.

(A) The functions $\omega_\pm$ are of classes $C^{\mu_\pm + 1}$ for some values $\mu_\pm = 2, 3,...,$ and the sets

$$\beta(\omega_\pm, \mu_\pm + 1) = \bigcap_{j=2}^{\mu_\pm + 1} \beta_j(\omega_\pm) = \varnothing \tag{2.19}$$

Here and below we denote

$$\beta_j(\omega) = \left\{\theta: \frac{d^j}{d\theta^j} \omega(\theta) = 0\right\}, \qquad j \geqslant 1 \tag{2.20}$$

The values $\mu_{\pm}$ are chosen for the remainder of the paper to be minimal among all numbers for which these conditions hold true.

For the original XY model, condition (A) is fulfilled iff at least two values among α, β, and h_0 are nonzero. The latter restriction appeared in Refs. 5 and 6 (where it is called the nondegeneracy condition).[4]

In this section we discuss the problem of convergence to a stationary state under the action of the dynamics in the generalized XY model. In connection with this, we introduce the following condition (B) on a ϕ-invariant state Q of $\mathfrak{M}$, which is formulated in terms of the even state ψ^{*-1} of $\mathfrak{U}^-$ ($\psi^{*-1}Q$ stands for the inverse image of Q under ψ^*):

(B) For any m, $n=0, 1,\ldots,$ with even $(m+n)>0$ there exists $d=d(m,n)>0$ such that

$$\lim_{s\to\infty} s^d \rho^{(m,n)}_{\psi^{*-1}Q}(s)=0 \tag{2.21}$$

Here

$$\begin{aligned}\rho^{(m,n)}_{\psi^{*-1}Q}(s)=&\sup_{s_1,s_2\in Z^1:s_1<s_2}\ \max_{m',n'}{}^{(m,n)}\ \sup{}^{(s_1,s_2;s)}_{(m',n';m,n)}\\ &\left|(\psi^{*-1}Q)\left(\prod_{p=1}^{m'} a^+_{j_p}\prod_{p'=1}^{m-m'} a^+_{j'_{p'}}\prod_{q'=1}^{n-n'} a_{k'_{q'}}\prod_{q=1}^{n'} a_{k_q}\right)\right.\\ &-(\psi^{*-1}Q)\left(\prod_{p=1}^{m'} a^+_{j_p}\prod_{q=1}^{n'} a_{k_q}\right)\\ &\left.\times(\psi^{*-1}Q)\left(\prod_{p'=1}^{m-m'} a^+_{j'_{p'}}\prod_{q'=1}^{n-n'} a_{k'_{q'}}\right)\right|\end{aligned} \tag{2.22}$$

and the maximum $\max^{(m,n)}_{m',n'}$ on the rhs of (2.22) is taken over all values $m'=0,\ldots,m$ and $n'=0,\ldots,n$ with $m'+n'\geqslant 1$, and the second supremum $\sup^{(s_1,s_2;s)}_{(m',n';m,n)}$ is taken over all ordered sets of integers $j_1,\ldots,j_{m'}$, $k_1,\ldots,k_{n'}\in[s_1,s_2]$ and $j'_1,\ldots,j'_{m-m'}$, $k'_1,\ldots,k'_{n-n'}\notin[s_1-s, s_2+s]$.

Condition (B) expresses the property of "decay of correlations" in the state $\psi^{*-1}Q$. Of course, this condition may be written in terms of the state Q itself. But in such a form it appears to be complicated. A stronger condition written in terms of Q is the following.

(B′) An element of the *-algebra $\mathfrak{M}^0$ is called a monomial if it is a product of Pauli matrices σ^x_j, σ^y_j, σ^z_j. Then, for some $d>0$

$$\text{(i)}\qquad \lim_{s\to\infty} s^d\alpha^{(1)}_Q(s)=0 \tag{2.23}$$

[4] Note that there is an error in Ref. 19: on p. 97, sixth line from the bottom. The condition $(\beta\alpha)^2+h_0^2\neq 0$ should be replaced by $(\beta\alpha)^2+[h_0(\alpha+\beta)]^2>0$.

where

$$\alpha_Q^{(1)}(s) = \sup_{s_1, s_2 \in Z^1 : s_1 < s_2} \sup_{A_1, A_2}^{(s_1, s_2; s)} |Q(A_1 A_2) - Q(A_1)\, Q(A_2)| \tag{2.24}$$

and the second supremun in (2.24) is taken over all monomials $A_1 \in \mathfrak{M}_{[s_1, s_2] \cap Z^1}$, $A_2 \in \mathfrak{M}_{Z^1 \setminus [s_1 - s, s_2 + s]}$; and

$$\text{(ii)} \qquad \lim_{s \to \infty} s^d \alpha_Q^{(2)}(s) = 0 \tag{2.25}$$

where

$$\alpha_Q^{(2)}(s) = \sup_{s_1, s_2 \in Z^1 : s_2 - s_1 = s} \sup_{A}^{(s_1, s_2)} |Q(A \sigma^z_{(s_1, s_2)})| \tag{2.26}$$

The second supremun in (2.26) is taken over all monomials $A \in \mathfrak{M}_{Z^1 \setminus (s_1, s_2)}$.

We are now able to formulate the theorem on convergence as $t \to \pm\infty$ for states $\tilde{W}_t^* Q$ defined by

$$\tilde{W}_t^* Q(A) = Q(\tilde{W}_{-t} A), \qquad A \in \mathfrak{M}$$

We call a ϕ-invariant state P of $\mathfrak{M}$ ψ-quasifree if its inverse image $\psi^{*-1} P$ is an even quasifree state of $\mathfrak{U}^-$. It is clear that a state P of $\mathfrak{M}$ is invariant with respect to the action of a group $\{\tilde{W}_t^* P \equiv P, t \in R^1)$ iff the state $\psi^{*-1} P$ is invariant with respect to the action of the corresponding group $\{\mathscr{T}_t\}$ $(\tilde{\mathscr{T}}_t^* \psi^{*-1} P \equiv \psi^{*-1} P, t \in R^1)$.

Theorem 2.1. Suppose that functions $\hat{g}^{(1)}$ and $\hat{g}^{(2)}$ that determine the group of *-automorphisms $\{\tilde{W}_t\}$ of $\mathfrak{M}$ satisfy condition (A) and that the initial ϕ-invariant state Q of $\mathfrak{M}$ satisfies condition (B). Then the states $\tilde{W}_t^* Q$ converge (in the w^*-topology) as $t \to \pm\infty$ to a limit ψ-quasifree state P iff

$$\begin{aligned} &\lim_{t \to \pm\infty} \tilde{W}_t^* Q(\sigma_j^{\delta_1} \sigma^z_{(j-1,k)} \sigma_k^{\delta_2}) \\ &\quad = P(\sigma_j^{\delta_1} \sigma^z_{(j-1,k)} \sigma_k^{\delta_2}), \qquad \delta_1, \delta_2 = \pm, j, k \in Z^1, \quad j \leqslant k \end{aligned} \tag{2.27}$$

Returning to the original *XY* model, we get an assertion on a convergence to a stationary (but in general nonequilibrium) state for the nondegenerate case (see above).

Theorem 2.1 reduces the problem of the convergence of states $\tilde{W}_t^* Q$ to the question of the convergence of their values on elements of a very special kind. The latter question will be solved separately. We shall give some sufficient conditions for the validity of the convergence in (2.27). The remarkable fact is that these conditions are formulated in terms of the values of the initial state Q on the same set of elements

$$Q(\sigma_j^{\delta_1}\sigma_{(j-1,k)}^z\sigma_k^{\delta_2}), \qquad \delta_1, \delta_2 = \pm, j, k \in Z^1, \quad j \leqslant k \tag{2.28}$$

This fact is explained by a linear connection between the values

$$\{\tilde{W}_t^* Q(\sigma_j^{\delta_1}\sigma_{(j-1,k)}^z\sigma_k^{\delta_2})\} \quad \text{and} \quad \{Q(\sigma_j^{\delta_1}\sigma_{(j-1,k)}^z\sigma_k^{\delta_2})\}$$

In view of this, the following construction will be done.

For fixed δ_1, $\delta_2 = \pm$, the values $Q(\sigma_j^{\delta_1}\sigma_{(j-1,k)}^z\sigma_k^{\delta_2})$ determine an infinite matrix $M_Q^{\delta_1,\delta_2}$, which generates a bounded linear operator in U. For $j \leqslant k$ we set the matrix elements $(M_Q^{\delta_1,\delta_2})_{j,k}$ to be equal to the values (2.28). Further, we set for $j > k$

$$(M_Q^{\delta,\delta})_{j,k} = -(M_Q^{\delta,\delta})_{k,j}, \qquad \delta = \pm \tag{2.29a}$$

$$(M_Q^{\delta_1,\delta_2})_{j,k} = (M_Q^{\delta_1,\delta_2})_{k,j}, \qquad \delta_1 \neq \delta_2 \tag{2.29b}$$

The operator under consideration is denoted by the same symbol $M_Q^{\delta_1,\delta_2}$. The quadruple of operators $M_Q^{\delta_1,\delta_2}$, $\delta_1, \delta_2 = \pm$, forms a 2×2 matrix

$$\mathbb{M}_Q = \begin{pmatrix} M_Q^{+,+} & M_Q^{+,-} \\ M_Q^{-,+} & M_Q^{-,-} \end{pmatrix}$$

Operators (or, equivalently, infinite matrices) $M_Q^{\delta_1,\delta_2}$, $\delta_1, \delta_2 = \pm$, satisfy, in addition to (2.29a) and (2.29b), a number of other conditions. Namely,

$$\begin{aligned} (M_Q^{+,-})_{j,k} + (M_Q^{-,+})_{k,j} &= 1, \qquad k = j \\ &= 0, \qquad k \neq j \end{aligned} \tag{2.29c}$$

Moreover, the norm $\|M_Q^{\delta_1,\delta_2}\| \leqslant 1$, and for all f, $g \in U$

$$\langle M_Q^{-,+} f, f\rangle + \langle M_Q^{+,-} g, g\rangle + \langle M_Q^{+,+} f, g\rangle + \langle M_Q^{-,-} g, f\rangle \geqslant 0 \tag{2.30}$$

It is not hard to check that for any operator 2×2 matrix $\mathbb{M}$ satisfying (2.29)–(2.30) one can find a state Q with $\mathbb{M}_Q = M$ (an example of such a state is the ψ-quasifree state determined by $\mathbb{M}$).

For later use (see Section 3), all the restrictions on $\mathbb{M}_Q$ listed above will be indicated as the condition (C) (imposed on an arbitrary operator matrix $\mathbb{M}$).

The matrices $\mathbb{M}_{\tilde{W}_t^* Q}$ and $\mathbb{M}_Q$, which correspond to states $\tilde{W}_t^* Q$ and Q, respectively, are related by

$$\mathbb{M}_{\tilde{W}_t^* Q} = \mathbb{J}_1 \tilde{\mathbb{T}}_{-t}^* \mathbb{J}_1 \mathbb{M}_Q \mathbb{J}_2 \tilde{\mathbb{T}}_{-t} \mathbb{J}_2 \tag{2.31}$$

where

$$\mathbb{J}_1 = \begin{pmatrix} J & 0 \\ 0 & E \end{pmatrix}, \qquad \mathbb{J}_2 = \begin{pmatrix} E & 0 \\ 0 & J \end{pmatrix}$$

and J denotes the complex conjugation in U.

Equality (2.31) leads to the following invariance condition:

$$\mathbb{J}_1(i\tilde{\mathbb{D}})^* \mathbb{J}_1 \mathbb{M}_Q + \mathbb{M}_Q \mathbb{J}_2(i\tilde{\mathbb{D}}) \mathbb{J}_2 = 0 \tag{2.32}$$

In case the operators $M_Q^{\delta_1,\delta_2}$ that constitute the matrix $\mathbb{M}_Q$ commute with the unitary space translation group $\{U_j\}$, it is convenient to pass to the Fourier transform. Here $\hat{M}_Q^{\delta_1,\delta_2}$ is the multiplication operator on a function $\hat{m}_Q^{\delta_1,\delta_2}$: $[-\pi, \pi) \to \mathbb{C}^1$. The functional 2×2 matrix constituted by these functions is denoted as $\hat{\mathbb{m}}_Q$. Equality (2.32) takes the form

$$\hat{\mathbb{F}}_1 \hat{\mathbb{m}}_Q + \hat{\mathbb{m}}_Q \hat{\mathbb{F}}_2 = 0 \tag{2.33}$$

where

$$\hat{\mathbb{F}}_1 = \begin{pmatrix} \mathrm{Ev}\, \hat{g}^{(2)} & \hat{g}^{(1)-} \\ \hat{g}^{(1)} & -\mathrm{Ev}\, \hat{g}^{(2)} \end{pmatrix}, \qquad \hat{\mathbb{F}}_2 = \begin{pmatrix} \mathrm{Ev}\, \hat{g}^{(2)} & -\hat{g}^{(1)} \\ -\hat{g}^{(1)-} & -\mathrm{Ev}\, \hat{g}^{(2)} \end{pmatrix} \tag{2.34}$$

Let us return to the validity of the relation (2.27). First, consider the simplest case, where the values (2.28) do not change if j, k are replaced by $j+m$, $k+m$, $m \in Z^1$ (this does not mean that the initial state Q is translationally invariant). In terms of the matrix $\mathbb{M}_Q$, this condition means that the operators $M_Q^{\delta_1,\delta_2}$ commute with the unitary space translation operators in U, i.e., appearing in the Fourier representation $\hat{M}_Q^{\delta_1,\delta_2}$ are the multiplication operators by functions $\hat{m}_Q^{\delta_1,\delta_2}$: $[-\pi, \pi) \to \mathbb{C}^1$. Consider the following condition on the functions $\hat{g}^{(1)}$, $\hat{g}^{(2)}$:

(A_1) For some value $\mu_0 = 1, 2,\ldots$, the function w [see (2.18b)] is of class C^{μ_0} and the set

$$\beta(w, \mu_0) = \bigcap_{j=1}^{\mu_0} \beta_j(w) = \varnothing \tag{2.35}$$

[see (2.20)].

In addition to the condition (A), the condition (A_1) is fulfilled for the original *XY* model iff at least two among the values α, β, and h_0 are different from zero.

Theorem 2.2. Suppose that $\hat{g}^{(1)}$ and $\hat{g}^{(2)}$ satisfy condition (A_1) and that initial operator matrices $\mathbb{M}_Q$ are constituted by the operators $M_Q^{\delta_1,\delta_2}$

commuting with the space translations. Then, for all $\delta_1, \delta_2 = \pm$ and $j, k \in Z^1$, $j \leqslant k$, the following limit exists:

$$\lim_{t \to \pm\infty} \tilde{W}_t^* Q(\sigma_j^{\delta_1} \sigma_{(j-1,k)}^z \sigma_k^{\delta_2}) = P(\sigma_j^{\delta_1} \sigma_{(j-1,k)}^z \sigma_k^{\delta_2}) \tag{2.36}$$

and is given by the value of a ψ-quasifree, translationally invariant and $\tilde{W}_t^*$-invariant state P. The state P is uniquely determined by the corresponding operator matrix $\mathbb{M}_P$, which is given by

$$\mathbb{M}_P = L\mathbb{M}_Q \tag{2.37}$$

where L denotes the linear projection onto the subspace of $\tilde{W}_t^*$-invariant matrices [see (2.32)]. In the Fourier transform

$$\hat{L}\hat{\mathbb{M}} = 1/2(\hat{\mathbb{M}} - \hat{\mathbb{W}}^{-1}\hat{\mathbb{F}}_1 \hat{\mathbb{M}} \hat{\mathbb{F}}_2 \hat{\mathbb{W}}^{-1}) \tag{2.38}$$

$$\hat{\mathbb{W}}^{-1} = \hat{W}^{-1}\hat{\mathbb{E}} \tag{2.39}$$

$\hat{W}^{-1}$ is the multiplication operator on the function w^{-1} [see (2.18b)] and $\hat{\mathbb{F}}_1$ and $\hat{\mathbb{F}}_2$ are defined in (2.34).

Returning to (2.14a) and (2.14b), one can write down the rhs of (2.38) for the original XY model explicitly. In particular, the equality becomes very simple in the case of the XX model ($\alpha = \beta$)

$$L\mathbb{M} = \mathbb{M}_{\hat{a}\mathrm{diag}} \tag{2.38'}$$

(here and below "adiag" indicates the off-diagonal part of a matrix). This reflects the fact that the corresponding infinitesimal matrix $\mathbb{D}$ is diagonal: in this case a translationally invariant quasifree state is $\mathscr{T}_t^*$-invariant iff it is gauge-invariant.

In a similar way one can consider initial states Q with "periodic" expectation values (2.28), i.e., with values that do not change if j, k are replaced by $j + ms$, $k + ms$, $m \in Z^1$, for some $s \in Z^1$. In this case, under some conditions on $\hat{g}^{(1)}$, $\hat{g}^{(2)}$ we can prove the convergence (2.36) with a ψ-quasifree and $\tilde{W}_t^*$-invariant, but not necessarily translationally invariant (in general, "periodic") state P. For the sake of brevity we shall not go into detail and consider here a more general case of initial states with "almost periodic" values (2.28) (which, however, will require more restrictive conditions on $\hat{g}^{(1)}$, $\hat{g}^{(2)}$).

Assume that the operators $M_Q^{\delta_1,\delta_2}$ in the Fourier representation are written in the form

$$\hat{M}_Q^{\delta_1,\delta_2} = \int_{[-\pi,\pi)} \nu_Q(d\lambda)\, \hat{M}_{Q,\lambda}^{\delta_1,\delta_2} \hat{S}_\lambda \tag{2.40}$$

where ν is a finite (Borel) measure on $[-\pi, \pi)$, $\hat{M}^{\delta_1,\delta_2}_{Q,\lambda}$ for every fixed λ is the operator of multiplication by the function $\hat{m}^{\delta_1,\delta_2}_{Q,\lambda}$: $[-\pi, \pi) \to \mathbb{C}^1$, and $\hat{S}_\lambda$ is the operator of the shift on the value λ [in $L_2([-\pi, \pi))$]. The functions $\hat{m}^{\delta_1,\delta_2}_{Q,\lambda}$ are assumed to be uniformly bounded $[\sup_{\lambda,\theta \in [-\pi,\pi)} |\hat{m}^{\delta_1,\delta_2}_{Q,\lambda}(\theta)| < \infty]$ and the measure ν is assumed to have a positive atom at the origin: $\nu(\{0\}) = \nu_0 > 0$. Under these conditions we say that Q is a state with the almost periodic expectation values (2.28).

Such a definition is motivated as follows. In the particular case when ν is the uniform distribution on the finite set$\{(2\pi l/s)(\mathrm{mod}[-\pi, \pi))$; $l = 0,..., s-1\}$ we get the periodic case mentioned above. If $s = 1$, i.e., if the measure ν is concentrated at the origin, we return to the case considered in Theorem 2.2.

We impose the following condition (A_2) on the functions $\hat{g}^{(1)}$, $\hat{g}^{(2)}$:

(A_2) For some value $\mu = 1, 2,...,$ the functions $\omega_\pm$ [see (2.18a), (2.18b)] are of class C^μ and for any nonzero $\lambda \in [-\pi, \pi)$ and δ_1, $\delta_2 = \pm$ the sets

$$\bar{\beta}(\omega_{\delta_1}(\cdot + \lambda) - \omega_{\delta_2}, \mu) = \bigcap_{j=1}^{\mu} \beta_j(\omega_{\delta_1}(\cdot + \lambda) + \omega_{\delta_2}) = \varnothing$$

[see (2.20)].

For the original *XY* model the condition (A_2) holds iff $h_0 \neq 0$ and at least one of the values α and β is nonzero.

Theorem 2.3. Let the functions $\hat{g}^{(1)}$, $\hat{g}^{(2)}$ satisfy condition (A_2). Then for any state Q with the almost periodic expectation values (2.28) and all δ_1, $\delta_2 = \pm$ and $j, k \in Z^1$, $j \leqslant k$, the limit (2.36) exists and coincides with the value of a ψ-quasifree, translationally invariant, $\tilde{W}^*_t$-invariant state P. This state is uniquely determined by

$$\hat{\mathbb{M}}_P = \nu_0 \hat{L} \hat{\mathbb{M}}_{Q,0}$$

where $\hat{L}$ is given by (2.38).

We now pass to the proofs of Theorems 2.1–2.3.

Proof of Theorem 2.1. By using the isomorphism ψ, one can reduce the problem of studying the states $\tilde{W}^*_t Q$ of the C^*-algebra $\mathfrak{M}$ to that of studying the states $\tilde{\mathcal{T}}^*_t \psi^{*-1} Q$ of the C^*-algebra $\mathfrak{U}^-$. The statement of Theorem 2.1 is an immediate corollary of the following proposition (see Ref. 17), Theorem 2.1).

Proposition 1. Let the functions $\hat{g}^{(1)}$, $\hat{g}^{(2)}$ determining the group of Bogoliubov transformations $\{\tilde{\mathcal{T}}_t\}$ on the C^*-algebra $\mathfrak{U}^-$ satisfy condition (A). Assume that an initial state Q' of $\mathfrak{U}^-$ is even and satisfies the condition (B). Then the states $\tilde{\mathscr{T}}_t^* Q'$ converge (in the w^*-topology) as $t \to \pm\infty$ to a $\tilde{\mathscr{T}}_t^*$-invariant, even, quasifree state iff for any $j, k \in Z^1$

$$\lim_{t \to \pm\infty} \tilde{\mathscr{T}}_t^* Q'(a_j^\# a_k) = P'(a_j^\# a_k) \tag{2.41}$$

Proof of Theorem 2.2. Again use the isomorphism ψ. This allows us to reduce the problem of convergence (2.36) for the expectation values (2.28) to the problem of convergence for the values $\tilde{\mathscr{T}}_t^* \psi^{*-1} Q(a_j^\# a_k)$. The values $\psi^{*-1} Q(a_j^\# a_k)$ depend on the differences $j-k$ only. It is convenient to call a state Q' of $\mathfrak{U}^-$ having this property a state with homogeneous expectation values

$$Q'(a_j^\# a_k), \qquad j, k \in Z^1 \tag{2.42}$$

The statement of Theorem 2.2 will follow from the following proposition (see Ref. 17, Section 2.5).

Proposition 2. Let the functions $\hat{g}^{(1)}, \hat{g}^{(2)}$ determining the group of Bogoliubov transformations $\{\tilde{\mathscr{T}}_t\}$ satisfy condition (A_1). Assume that Q' is a state of $\mathfrak{U}^-$ with homogeneous expectation values (2.42). Then, for any j, $k \in Z^1$, the limits

$$\lim_{t \to \pm\infty} \tilde{\mathscr{T}}_t^* Q'(a_j^\# a_k) \tag{2.43}$$

exist and correspond to a quasifree, translation-invariant, $\tilde{\mathscr{T}}_t^*$-invariant state P'.

In the same way, the statement of Theorem 2.3 follows from the following proposition (see Ref. 17, Section 2.7). In analogy with the above definitions, one introduces the notion of a state of the C^*-algebra $\mathfrak{U}^-$ with almost periodic values (2.42).

Proposition 3. Let the functions $\hat{g}^{(1)}, \hat{g}^{(2)}$ determining the group of Bogoliubov transformations $\{\tilde{\mathscr{T}}_t\}$ satisfy condition (A_2). Let Q' be a state of $\mathfrak{U}^-$ with almost period expectation values (2.42). Then, for any j, $k \in Z^1$, the limits (2.43) exist and correspond to a quasifree, translation-invariant, $\tilde{\mathscr{T}}_t^*$-invariant state P'.

3. TIME EVOLUTION OF LOCAL PARAMETERS IN THE HYDRODYNAMIC LIMIT FOR THE *XY* MODEL AND ITS GENERALIZATIONS

It is convenient to denote by $I(y, u)$ the interval $[y - u/2, y + u/2)$. We suppose that a group of $*$-automorphisms $\tilde{W}_t \colon \mathfrak{M}(+) \to \mathfrak{M}(+)$, $t \in R^1$, is fixed, which defines the dynamics for a generalized XY model (see the preceding section).

The group $\{\tilde{W}_t\}$ is determined by a constant h_0 and functions $f^{(1)},..., f^{(4)}$, or, equivalently, by functions $g^{(1)}$ and $g^{(2)}$, which are connected with $f^{(1)},..., f^{(4)}$ by Eqs. (2.17a) and (2.17b). We shall assume that the functions $g^{(1)}$ and $g^{(2)}$ satisfy condition (A) of Section 2. In what follows it is convenient to denote by μ the maximal value of the numbers $\mu_\pm$ figuring in this condition [recall that $\mu_\pm$ are chosen to be minimal numbers for which (2.19) holds].

In addition, we assume that the function w given by (2.18b) satisfies the conditions (A_1) of Section 2.

Let a family $\{M^{\delta_1,\delta_2}(x), x \in R^1, \delta_1, \delta_2 = \pm\}$ be given, where $M^{\delta_1,\delta_2}(x)$ is a bounded, linear operator in U that commutes with the unitary space translation operators. It is convenient to assume that $M^{\delta_1,\delta_2}(x)$ satisfies, for all x and δ_1, δ_2, condition (C) (see Section 2). We shall assume as well that the following condition holds:

D. The operators $M^{\delta_1,\delta_2}(x)$ depend on x in a smooth way (in the uniform operator topology), and the derivative $(\partial/\partial x)\, M^{\delta_1,\delta_2}(x)$ is a bounded operator with the norm $\|(\partial/\partial x)\, M^{\delta_1,\delta_2}(x)\|$, which is bounded uniformly in x within any bounded interval of R^1.

The image $\hat{M}^{\delta_1,\delta_2}(x)$ of $M^{\delta_1,\delta_2}(x)$ under Fourier transform is the operator of multiplication by a function $\hat{m}^{\delta_1,\delta_2}(x, \cdot)$: $[-\pi, \pi) \to \mathbb{C}^1$ with $\sup_{\theta \in [-\pi,\pi)} |\hat{m}^{\delta_1,\delta_2}(x, \theta)| \leqslant 1$. The condition (D) implies that (1) for almost all $\theta \in [-\pi, \pi)$ the derivative $(\partial/\partial x)\, \hat{m}^{\delta_1,\delta_2}(x, \theta)$ exists, and the supremum $\operatorname{ess\,sup} |(\partial/\partial x)\, \hat{m}^{\delta_1,\delta_2}(x, \theta)|$ is bounded uniformly in x within any bounded interval, and (2) for any $x \in R^1$

$$\lim_{\Delta \to 0} \sup_{\theta \in [-\pi,\pi)} |\Delta^{-1}(\hat{m}^{\delta_1,\delta_2}(x + \Delta, \theta) - \hat{m}^{\delta_1,\delta_2}(x, \theta)) \\ - (\partial/\partial x)\, \hat{m}^{\delta_1,\delta_2}(x, \theta)| = 0$$

The further conditions may be written in various versions; this will imply some differences in the formulation of the results. We first give one such version [conditions (E)–(G) below] and formulate the corresponding theorem (Theorem 3.1). Then we give another version [conditions (E*)–(G*) below] and formulate the corresponding theorem (Theorem 3.1*). After this we briefly discuss the difference between the two results.

We continue with the condition that the matrix elements $(M^{\delta_1,\delta_2}(x))_{j,k}$ (in the canonical basis of U) decay sufficiently rapidly. More precisely [cf. the conditions (B), (B′) of Section 2] we have:

(E) For some $d_1 \geqslant \min\{\mu, 3\}$

$$\lim_{s \to \infty} s^{d_1} \xi^{(1)}(s) = 0 \tag{3.1}$$

where

$$\xi^{(1)}(s) = \sup_{x \in R^1} \sup_{j,k \in Z^1:\, |j-k| \geqslant s} |(M^{\delta_1,\delta_2}(x))_{j,k}| \tag{3.2}$$

Finally, we suppose that a family of ϕ-invariant states $\{Q^\varepsilon, \varepsilon > 0\}$ on $\mathfrak{M}$ is given that satisfies the following two conditions:

(F) For some $\bar{d}_1 \geqslant \min\{\mu, 3\}$

$$\lim_{s \to \infty} s^{\bar{d}_1} \bar{\xi}^{(1)}(s) = 0 \tag{3.3}$$

where

$$\bar{\xi}^{(1)}(s) = \sup_{\varepsilon > 0} \max_{\delta_1,\delta_2 = \pm} \max_{\substack{j,k \in Z^1:\\ |j-k| \geqslant s}} |(M_{Q^\varepsilon}^{\delta_1,\delta_2})_{j,k}| \tag{3.4}$$

Condition (F) is weaker than the conditions (B) and (B′) of Section 2 (because now only $m+n=2$ is taken into account in (2.21)]. However, a rapid decay of pair correlations should be valid uniformly in ε.

(G) For every $\varepsilon > 0$ there exists a even integer $N_\varepsilon > 0$ with the following properties.

(i) The following condition holds:

$$\lim_{\varepsilon \to 0} \varepsilon\, |\ln \varepsilon|\, N_\varepsilon = 0 \tag{3.5a}$$

(ii) For some $\gamma \in [(\mu+1)^{-1}\mu, 1)$

$$\lim_{\varepsilon \to 0} \varepsilon^\gamma N_\varepsilon = \infty \tag{3.5b}$$

(iii) For all $v \in Z^1$ and all integers s_1, s_2 from the interval $I(vN_\varepsilon, 1/2N_\varepsilon)$

$$\begin{aligned} &|(M_{Q^\varepsilon}^{\delta_1,\delta_2})_{s_1,s_2} - (M^{\delta_1,\delta_2}(\varepsilon v N_\varepsilon))_{s_1,s_2}| \\ &\quad \leqslant \psi_1(|s_1 - s_2|)\, \psi_2(\mathrm{dist}(\{s_1, s_2\}, R^1 \backslash I(vN_\varepsilon, 1/2N_\varepsilon))) \end{aligned} \tag{3.6}$$

where $\psi_1, \psi_2\colon R_+^1 \to R_+^1$ are decreasing L_1-functions.

Notice that from condition (G) it follows that for all $x \in R^1$ and s, $s' \in Z^1$

$$\begin{aligned} &\lim_{\varepsilon \to 0} (M_{Q^\varepsilon}^{\delta_1,\delta_2})_{n(\varepsilon,x)+s,n(\varepsilon,x)+s'} \\ &\qquad = (M^{\delta_1,\delta_2}(x))_{s,s'}, \qquad \delta_1, \delta_2 = \pm \end{aligned} \tag{3.7}$$

where

$$n(\varepsilon, x) = [[\varepsilon^{-1}x]\, N_\varepsilon^{-1}]\, N_\varepsilon \qquad (3.7')$$

Moreover, given $x \in R^1$, the convergence in (3.7) is uniform with respect to s, s' in any bounded interval (and even in an interval $[n(\varepsilon, x) - o(\varepsilon^{-1}), n(\varepsilon, x) + o(\varepsilon^{-1})]$). Physically speaking, the family of operators $\{M^{\delta_1,\delta_2}(x), x \in R^1, \delta_1, \delta_2 = \pm\}$ determines the macroscopic spatial "profile" of local parameters that characterize the states Q^ε, $\varepsilon > 0$. The role of these parameters is played by the expectation values (2.28). The value ε indicates the "typical" ratio of micro- and macroscales in space and time.

Condition (G) [as well as condition (G*) below] expresses the property of "hydrodynamic stability" of the family $\{Q^\varepsilon\}$ at time 0. The problem of the hydrodynamic description of the evolution involves, in particular, the verification of the hydrodynamic stability at a macroscopic time $t \neq 0$.

Theorem 3.1. Assume that the group $\{\tilde{W}_t\}$, the family of operators $\{M^{\delta_1,\delta_2}(x), x \in R^1, \delta_1, \delta_2 = \pm\}$, and the family of states $\{Q^\varepsilon, \varepsilon > 0\}$ satisfy the conditions (A), (A_1), and (C)–(G) formulated above. Then for all $x \in R^1$, $s, s' \in Z^1$, and nonzero $t \in R^1$ the following limit exists:

$$\begin{aligned} &\lim_{\varepsilon \to 0} (M^{\delta_1,\delta_2}_{\tilde{W}^*_{\varepsilon^{-1}t}Q^\varepsilon})_{n(\varepsilon,x)+s, n(\varepsilon,x)+s'} \\ &\quad = (M^{\delta_1,\delta_2}(t; x))_{s,s'}, \qquad \delta_1, \delta_2 = \pm \end{aligned} \qquad (3.8)$$

The limiting values (3.8) determine the operators $M^{\delta_1,\delta_2}(t; x)$ with $(M^{\delta_1,\delta_2}(t; x))_{s_1,s_2} = (M^{\delta_1,\delta_2}(t; x))_{0,s_2-s_1}$ (i.e., the operators that commute with the unitary operators of space shifts). Moreover, the operators $M^{\delta_1,\delta_2}(t; x)$ satisfy the invariance equation (2.32).

The second version of the restrictions consists of the following conditions:

(E*) The relation (3.1) is valid for any $d_1 > 0$.

(F*) The relation (3.3) holds for any $\bar{d}_1 > 0$.

(G*) For every $\varepsilon > 0$ there exists an even integer $N_\varepsilon > 0$ with the following properties:

(i) $\lim_{\varepsilon \to 0} \varepsilon^{\gamma_0} N_\varepsilon = 0$, $\lim_{\varepsilon \to 0} \varepsilon^{\gamma} N_\varepsilon = \infty$ for some γ_0 and γ that obey $(\mu + 1)^{-1}\, \mu \leqslant \gamma < \gamma_0 < 1$.

(ii) For all $u \in R^1$ and integers $s, s' \in I(uN_\varepsilon, 1/2N_\varepsilon)$

$$\begin{aligned} &|(M^{\delta_1,\delta_2}_{Q^\varepsilon})_{s,s'} - (M^{\delta_1,\delta_2}(\varepsilon u N_\varepsilon))_{s,s'}| \\ &\quad < \min\{\psi(|s - s'|), c_0\varepsilon\, |s - uN_\varepsilon|, c_0\varepsilon\, |s' - uN_\varepsilon|\} \end{aligned} \qquad (3.6^*)$$

where $c_0 > 0$ is a constant, and ψ is a monotonic, nonnegative function such that $\lim_{s\to\infty} s^d\psi(s) = 0$ for any $d > 0$.

Theorem 3.1.*. Assume that the group $\{\tilde{W}_t\}$, the family of operators $\{M^{\delta_1,\delta_2}(x)\}$, and the family of states $\{Q^\varepsilon, \varepsilon > 0\}$ satisfy the conditions (A), (A_1), (C), (D), and (E*)–(G*). Then, for all $x \in R^1$, $s, s' \in Z^1$, and nonzero $t \in R^1$ the following limit exists

$$\lim_{\varepsilon\to 0} (M^{\delta_1,\delta_2}_{\tilde{W}^*_{\varepsilon^{-1}t}Q^\varepsilon})_{[\varepsilon^{-1}x]+s,[\varepsilon^{-1}x]+s'} = (M^{\delta_1,\delta_2}(t;x))_{s,s'}, \qquad \delta_1, \delta_2 = \pm \tag{3.8*}$$

The limiting values (3.8*) coincide with (3.8).

The formal difference between Theorems 3.1 and 3.1* is that $n(\varepsilon, x)$ is replaced by more "plausible" space scaling $[\varepsilon^{-1}x]$. However, the difference between (G) and (G*) [more precisely, between (3.6) and (3.6*)] is more serious. The assumption (G) means, roughly speaking, that the state Q^ε is "almost homogeneous" on any interval $I(vN_\varepsilon, 1/2N_\varepsilon)$, $v \in Z^1$, whereas (G*) says that Q^ε is "slowly varying" on Z^1. For verifying conditions of both theorems one uses the theory of Gibbs states (see Section 4).

The family of matrices $\{\mathbb{M}(t;x), x \in R^1, t \in R^1\backslash\{0\}\}$ constituted by the operators $M^{\delta_1,\delta_2}(t;x)$ may be computed via the initial family of matrices $\{\mathbb{M}(x), x \in R^1\}$ made up of the operators $M^{\delta_1,\delta_2}(x)$. Let $\hat{\mathbb{m}}(t;x)$ and $\hat{\mathbb{m}}(x)$ denote the functional 2×2 matrices that correspond to $\mathbb{M}(t;x)$ and $\mathbb{M}(x)$, respectively, after taking the Fourier transform [recall that the functions $\hat{m}^{\delta_1,\delta_2}(t;x,\cdot)$ and $\hat{m}^{\delta_1,\delta_2}(x,\cdot)$, which make up the matrices $\hat{\mathbb{m}}(t;x)$ and $\hat{\mathbb{m}}(x)$, are defined on $[-\pi,\pi)$]. Consider the functional matrices $\hat{\mathbb{m}}_t(x,\delta)$, $x, t \in R^1$, $\delta = \pm$, given by

$$\hat{\mathbb{m}}_t(x,\delta) = \begin{pmatrix} \hat{m}^{+,+}(x-\omega'_\delta(\theta)\,t,\theta) & \hat{m}^{+,-}(x-\omega'_\delta(\theta)\,t,\theta) \\ \hat{m}^{-,+}(x-\omega'_\delta(\theta)\,t,\theta) & \hat{m}^{-,-}(x-\omega'_\delta(\theta)\,t,\theta) \end{pmatrix}, \quad \theta \in [-\pi,\pi) \tag{3.9}$$

Theorem 3.2. The matrices $\hat{\mathbb{m}}(t;x)$, $x \in R^1$, $t \in R^1\backslash\{0\}$, are written in the form

$$\hat{\mathbb{m}}(t;x) = 1/2[\hat{\mathbb{m}}_+(t;x) + \hat{\mathbb{m}}_-(t;x)] \tag{3.10a}$$

where

$$\begin{aligned} \hat{\mathbb{m}}_+(t;x) = 1/2[&\hat{\mathbb{m}}_t(x,+) - \hat{\mathbb{W}}^{-1}\hat{\mathbb{F}}_1\hat{\mathbb{I}}\hat{\mathbb{m}}_t(x,+) \\ &+ \hat{\mathbb{I}}\hat{\mathbb{m}}_t(x,+)\,\hat{\mathbb{F}}_2\hat{\mathbb{W}}^{-1} - \hat{\mathbb{W}}^{-1}\hat{\mathbb{F}}_1\hat{\mathbb{m}}_t(x,+)\,\hat{\mathbb{F}}_2\hat{\mathbb{W}}^{-1}] \end{aligned} \tag{3.10b}$$

$$\begin{aligned} \hat{\mathbb{m}}_-(t;x) = 1/2[&\hat{\mathbb{m}}_t(x,-) + \hat{\mathbb{W}}^{-1}\hat{\mathbb{F}}_1\hat{\mathbb{I}}\hat{\mathbb{m}}_t(x,-) \\ &- \hat{\mathbb{I}}\hat{\mathbb{m}}_t(x,-)\,\hat{\mathbb{F}}_2\hat{\mathbb{W}}^{-1} - \hat{\mathbb{W}}^{-1}\hat{\mathbb{F}}_1\hat{\mathbb{m}}_t(x,-)\,\hat{\mathbb{F}}_2\hat{\mathbb{W}}^{-1}] \end{aligned} \tag{3.10c}$$

Here

$$\hat{\mathbb{I}} = \begin{pmatrix} -\hat{E} & 0 \\ 0 & \hat{E} \end{pmatrix}$$

where $\hat{E}$ is the unit operator in $\hat{U}$ and the matrices $\hat{\mathbb{F}}_1$, $\hat{\mathbb{F}}_2$, and $\hat{\mathbb{W}}^{-1}$ are defined in Section 2 [see (2.34), (2.39)]. If the matrices $\hat{\mathbb{m}}(x)$ satisfy the invariance equation (2.35), then the formulas (3.11b) and (3.11c) can be simplified:

$$\hat{\mathbb{m}}_{\pm}(t; x) = \hat{\mathbb{m}}_t(x, \pm) \pm \hat{\mathbb{B}}_1 \hat{\mathbb{W}}^{-1} \hat{\mathbb{m}}_t(x, \pm) \tag{3.11}$$

Here $\hat{\mathbb{B}}_1 = \hat{B}_1 \hat{\mathbb{E}}$, where $\hat{B}_1$ is the operator of multiplication by the function $\operatorname{Ev} \hat{g}^{(2)}$, and $\hat{\mathbb{E}}$ is the unit matrix.

Proof of Theorems 3.1, 3.1, and 3.2.* As in the preceding section, we use the *-automorphism ψ. Then we obtain the family of even states $\{\psi^{*-1} Q^{\varepsilon}, \varepsilon > 0\}$ of the C^*-algebra $\mathfrak{U}^-$. The following result proven in Ref. 23. (see Ref. 23, Theorems 3.1 and 3.1′) is nothing but the reformulation of Theorems 3.1, 3.1*, and 3.2.

Proposition 4. Let the functions $g^{(1)}$ and $g^{(2)}$ determining the group of Bogoliubov transformations $\{\tilde{\mathcal{T}}_t\}$ on the C^*-algebra $\mathfrak{U}^-$ satisfy the conditions (A) and (A_1) of Section 2. Assume that the initial family of the operator matrices $\{\mathbb{M}(x), x \in R^1\}$ is given, which is constituted by the operators $M^{\delta_1, \delta_2}(x)$, $\delta_1, \delta_2 = \pm$, satisfying conditions (C) and (D) of this section. Let $\{(Q')^{\varepsilon}, \varepsilon > 0\}$ be a family of even states of the C^*-algebra $\mathfrak{U}^-$ which satisfy either the conditions (E)–(G) or the conditions (E*)–(G*) [replacing Q^{ε} by $(Q')^{\varepsilon}$]. Then, for any $x, t \in R^1$, $t \neq 0$, and $s, s' \in Z^1$, the limits (3.8) [respectively, (3.8*)] exist, and the operator matrices $\mathbb{M}(t; x)$ made up of the limit operators $M^{\delta_1, \delta_2}(t; x)$, $\delta_1, \delta_1 = \pm$, are defined by (3.10)–(3.11).

The evolution of the spatial profile of the local parameters as given by (3.10)–(3.11) may be described by means of a system of differential equations (which plays, for the generalized XY model, the role of the Euler system of equations). In the general case this system has a complicated form, which simplifies somewhat if one assumes that the initial family $\{\hat{\mathbb{m}}(x)\}$ consists of matrices satisfying Eq. (2.33). In this case the family $\{\hat{\mathbb{m}}(t; x)\}$ may be described as the solution of the following Cauchy problem:

$$\frac{\partial^2}{\partial t^2} \hat{\mathbb{m}}(t; x) + (\omega'_+ + \omega'_-) \frac{\partial^2}{\partial t\, \partial x} \hat{\mathbb{m}}(t; x) + \omega'_+ \omega'_- \frac{\partial^2}{\partial x^2} \hat{\mathbb{m}}(t; x) = 0 \tag{3.12}$$

$$\hat{\mathbb{m}}(0; 0) = \hat{\mathbb{m}}(x) \tag{3.13a}$$

$$\frac{\partial}{\partial t}\hat{\mathfrak{m}}(t;x)|_{t=0}$$

$$= -\frac{1}{2}\left[\left(1+\frac{\text{Ev}\,\hat{g}^{(2)}}{w}\right)\omega'_{+}+\left(1-\frac{\text{Ev}\,\hat{g}^{(2)}}{w}\right)\omega'_{-}\right]\frac{\partial}{\partial x}\hat{\mathfrak{m}}(x) \qquad (3.13b)$$

The main feature of the problem (3.12), (3.13a), and (3.13b) is that, given $\theta \in [-\pi, \pi)$, the family of (complex) 2×2 matrices $\{\hat{\mathfrak{m}}(t; x, \theta), t, x \in R^1\}$ is varying in space and time "independently" of other families. Physically speaking, one gets an independent evolution of various "normal (matrix) modes" indexed by points of the circumference $[-\pi, \pi)$.

For the original XY model Eq. (3.12) takes the form

$$\frac{\partial^2}{\partial t^2}\hat{\mathfrak{m}}(t;x)-4[h_0(\alpha+\beta)\sin\theta-2\alpha\beta\sin 2\theta]^2$$

$$\times\{[h_0-(\alpha+\beta)\cos\theta]^2+(\alpha-\beta)^2\sin^2\theta\}^{-1}\frac{\partial^2}{\partial x^2}\hat{\mathfrak{m}}(t;x)=0 \qquad (3.14)$$

Finally, for the XX model the matrix $\hat{\mathfrak{m}}_{-}(t;x)=0$ and only one term remains in the formula (3.10a). This is the antidiagonal matrix $\hat{\mathfrak{m}}_{+}(t;x)$. Here the hydrodynamic equation looks very simple even without the assumption that the initial matrices $\hat{\mathfrak{m}}(x)$ satisfy (2.33). Namely, in this case one gets the equation

$$\frac{\partial}{\partial t}\hat{\mathfrak{m}}(t;x)=-4\alpha\sin\theta\frac{\partial}{\partial x}\hat{\mathfrak{m}}(t;x) \qquad (3.15)$$

with the Cauchy date

$$\hat{\mathfrak{m}}(0;x)=\hat{L}\hat{\mathfrak{m}}(x)=\hat{\mathfrak{m}}_{\text{adiag}}(x) \qquad (3.16)$$

[cf. (2.38′)]. This is the transfer equation, which describes the independent motion of normal modes indexed by the points of $[-\pi, \pi)$. The value $4\alpha\sin\theta$ is the "velocity" of the motion for the mode labeled by θ. Notice that the value h_0 does not appear in the hydrodynamic description of the XX model: this reflects the well-known fact that the normal mode velocity is given by the gradient of the energy density, which is given here by $2(h_0-2\alpha\cos\theta)$.

4. EXAMPLES OF STATES SATISFYING THE CONDITIONS OF SECTIONS 2 AND 3

In this section we give examples of initial states Q and families of initial states $\{Q^\varepsilon\}$ of the C^*-algebra $\mathfrak{M}$ for which the conditions of the above theorems are fulfilled. The simplest way to construct such examples is to consider ψ-quasifree states with given matrices $\mathbb{M}_Q$ [see (2.30)]. However, a larger (and physically more natural) class of examples may be given by means of the theory of Gibbs states (KMS states) (Ref. 12, Chapters 5.3, 5.4, 6.1, 6.2).

For the sake of brevity we state the results without proofs. The proofs require a modification of well-known constructions used in the references cited below that is rather simple in principle, but quite complicated on a technical level.

In the process of constructing KMS states one usually starts with a derivation Θ on the C^*-algebra $\mathfrak{M}$ which, formally speaking, may be written as

$$\Theta A = i[h, A] \tag{4.1}$$

[cf. (2.2), (2.8)], where the (formal) Hamiltonian h is the infinite sum

$$h = \sum_{j \in Z^1} h^{(j)} \tag{4.2}$$

of ϕ-invariant Hermitian elements $h^{(j)} \in \mathfrak{M}(h^{(j)*} = h^{(j)})$, $j \in Z^1$ (the condition of ϕ-invariance is supposed to hold in what follows without repeating it every time).

In particular, the translationally invariant Hamiltonians are written as

$$h = \sum_{j \in Z^1} U_j b \tag{4.3}$$

where b is a fixed Hermitian element of $\mathfrak{M}$. Among the translationally noninvariant Hamiltonians, the simplest ones are perhaps periodic Hamiltonians, which are characterized by the following condition: there exists a positive integer k and a collection of Hermitian elements $b_0,..., b_{k-1} \in \mathfrak{M}$ such that

$$h^{(j)} = U_j b_s \quad \text{if} \quad j = s (\text{mod}\, k), \quad j \in Z^1, \quad s = 0,..., k-1 \tag{4.4}$$

The smallest number k with this property is called the period of the Hamiltonian h.

Translationally invariant and periodic Hamiltonians lead to translationally invariant and periodic derivations, respectively [$U_j \Theta A = \Theta U_j A$, $j \in Z^1$, and $U_j \Theta A = \Theta U_j A$, $j = 0$ (mod k)].

The problem of the existence and uniqueness of KMS states of the C^*-algebra $\mathfrak{M}$ that correspond to translationally invariant Hamiltonians (4.3), in the case of a local element $b \in \mathfrak{M}^0$, has been investigated in a series of papers[25 27] initiated by the fundamental results by H. Araki.[25] The existence and uniqueness of KMS states were later proven for the case where the element $b \in \mathfrak{M}$ is approximated "sufficiently rapidly" by elements of the local *-algebra $\mathfrak{M}^0$.[28,29]

The KMS states constructed in Refs. 25, 28, and 29 are ϕ-invariant (because the element b is ϕ-invariant), translationally invariant, and satisfy condition (B′), part (i) on the decay of correlations [see (2.23)–(2.24)].

Analysis of the proofs in Refs. 25–29 shows that the existence and uniqueness of KMS states also occur for periodic Hamiltonians whenever the corresponding "generating elements" $b_0,..., b_{k-1}$ are either from the *-algebra $\mathfrak{M}^0$ or a approximated sufficiently rapidly by local elements. Moreover, the KMS states arising here are ϕ-invariant and have good properties of the decay of correlations.

However, for verifying the conditions (B) (see Section 2) and (F) and (F*) (see Section 3) we need to pass to the C^*-algebra $\mathfrak{U}^-$. From this point of view it is convenient to investigate directly the corresponding states of $\mathfrak{U}^-$. Hence, we restrict ourselves to studying the Hamiltonians h of the form (4.2) with

$$h^{(j)} = (\mu_j + 1)\, \sigma_j^+ \sigma_j^- - 1/2(\sigma_j^+ \sigma_j^z \sigma_{j+1}^- - \sigma_j^- \sigma_j^z \sigma_{j+1}^+) \\ + \sum_{j':j' \geqslant j} V(j'-j)\, \sigma_j^+ \sigma_j^- \sigma_{j'}^+ \sigma_{j'}^- \tag{4.5}$$

where $\{\mu_j\}$ is a periodic ($\mu_{j+km} = \mu_j$, $j, m \in Z^1$) sequence of reals with the period k (if $k=1$, then $\mu_j = \mu_0$), and V is a real-valued function on the set of nonnegative integers with a finite support [$V(j) = 0$ for $j \geqslant r_0$]. After applying the *-isomorphism ψ, we have that the Hamiltonian h corresponds to the (formal) Hamiltonian

$$g = -\frac{1}{2} \sum_{j \in Z^1} a_j^+ (\Delta a)_j + \sum_{j \in Z^1} \mu_j a_j^+ a_j \\ + \sum_{j,j':j' \geqslant j} V(j'-j)\, a_j^+ a_j a_{j'}^+ a_{j'} \tag{4.6}$$

This correspondence may be extended onto KMS states: the (unique) KMS state Q with respect to the group of *-automorphisms of the C^*-algebra $\mathfrak{M}$ generated by the derivation Θ of the form (4.1), (4.2), (4.5) has the inverse image $\psi^{*-1}Q$, the KMS state with respect to the group of *-automorphisms of the C^*-algebra $\mathfrak{U}^-$ generated by the derivation

$$\Psi A = i[g, A] \tag{4.7}$$

[cf. (2.8), (2.9)], where g is of the form (4.6), (4.7) [notice that $\psi^{*-1}Q$ is the unique KMS state for this group].

We note that the study of KMS states with respect to the derivation (4.6), (4.7) may be done without using the isomorphism ψ, by means of the methods developed in Refs. 30–35. More precisely, by using these methods, one can prove the following assertion.

Theorem 4.1. Let $\{\mu_j\}$ be a periodic sequence, and $V: Z^1_+ \to R^1$ be a function with finite support. Then there exists a unique KMS state G with respect to the group of *-automorphisms of $\mathfrak{U}^-$ generated by the derivation Ψ [see (4.6), (4.7)]. This is the state with periodic expectation values (2.42), which satisfies the condition (2.21) for any $d>0$ [in fact, $\rho_G^{(m,n)}(s)$ decays exponentially as $s\to\infty$]. Moreover, for any given $c_1, c_2 \in R^1$ $(c_1 \leqslant c_2)$, the quantity $\rho_G^{(m,n)}(s)$ decays uniformly for all KMS states $\tilde{G}$ corresponding to sequences $\{\mu_j\}$ with $c_1 \leqslant \mu_j \leqslant c_2$ (and the fixed function V). If $\{\mu_j\}$ is a translationally invariant sequence, then G is the translationally invariant state.

Theorem 4.1 indicates the class of states on the C^*-algebra $\mathfrak{M}$ for which one can check the condition (B), the main assumption (on the initial state) figuring in Theorem 2.1. This is the class of KMS states with respect to the groups generated by derivations Θ of the form (4.1), (4.2), (4.5). Furhtermore, Theorem 4.1 solves the question of verifying the relation (2.26).

To check the assumptions of Theorems 3.1 and 3.1* we have to consider a family of Hamiltonians $\{h_\varepsilon, \varepsilon>0\}$ of the form (4.2), (4.5), or, equivalently, the corresponding family $\{g_\varepsilon, \varepsilon>0\}$ of the form (4.7), where the sequence $\{\mu_j\}$ depends on ε: $\mu_j = \mu_j^\varepsilon$, $j\in Z^1$. More precisely, fix a smooth, periodic function $\lambda(x)$, $x\in R^1$, with the period $u>0$ and consider one of the two versions:

(a) Fix a family of even integers N_ε, $\varepsilon>0$, satisfying (3.5a), (3.5b) and set

$$\mu_j^\varepsilon = \lambda(vu[\varepsilon^{-1}N_\varepsilon^{-1}u]^{-1}) \qquad \text{if} \quad j\in I(vN_\varepsilon, 1/2N_\varepsilon) \tag{4.8}$$

for some $v\in Z^1$

$$\text{(a*)} \quad \mu_j^\varepsilon = \lambda(u[\varepsilon^{-1}u]^{-1} j), \qquad j\in Z^1. \tag{4.8*}$$

In both cases $\{\mu_j^\varepsilon, j\in Z^1\}$ is a periodic sequence for any $\varepsilon>0$.

Let G_x, $x\in R^1$, denote the KMS state with respect to the group of *-automorphisms of $\mathfrak{U}^-$ generated by the derivation Ψ of the form (4.6), (4.7) with $\mu_j = \lambda(x)$. Then the corresponding state $Q_x = \psi^* G_x$ is KMS with respect to the group of *-automorphisms of $\mathfrak{M}$ generated by the derivation Θ of the form (4.1), (4.2), (4.5). We set

$$M^{\delta_1,\delta_2}(x) = M_{Q_x}^{\delta_1,\delta_2}, \qquad \delta_1, \delta_2 = \pm \tag{4.9}$$

Since the states G_x (as well as Q_x) are translationally invariant, the operators $M^{\delta_1,\delta_2}(x)$ commute with the unitary operators of the space shifts. By construction, $M^{\delta_1,\delta_2}(x)$ satisfy the condition (C) (see Section 2). Notice (although this is not used below) that these operators are periodic in x.

By means of the methods developed in Refs. 25–29 and in Refs. 30–35 it is not hard to prove the following result.

Theorem 4.2. The family of operators $\{M^{\delta_1,\delta_2}(x), x \in R^1, \delta_1, \delta_2 = \pm\}$ satisfies the conditions (C), (D), and (E*) [and, hence, (E)]. In fact, the quantity $\xi^{(1)}(s)$ decays exponentially as $s \to \infty$.

Now we take the states $Q^\varepsilon = \psi^* G^\varepsilon$, $\varepsilon > 0$, where G^ε is the KMS state for the group of *-automorphisms of $\mathfrak{U}^-$ generated by the derivation Ψ^ε of the form (4.7) where the periodic Hamiltonian g_ε is given by (4.6) with μ_j^ε of the form either (4.8) or (4.8*). The equivalent definition is that Q^ε is the KMS state for the group of automorphisms of $\mathfrak{M}$ generated by the derivation Θ^ε of the form (4.1) where the periodic Hamiltonian h_ε is given by (4.2), (4.5) with the same μ_j^ε. As mentioned above, the existence and uniqueness of these states may be proven by using the methods of Refs. 25–35.

Theorem 4.3. The family of states $\{Q^\varepsilon, \varepsilon > 0\}$ satisfies the conditions (F), (G) [respectively, (F*), (G*)]. Thereby, the families of operators $\{M^{\delta_1,\delta_2}(x), x \in R^1, \delta_1, \delta_2 = \pm\}$ and states $\{Q^\varepsilon, \varepsilon > 0\}$ satisfy the assumptions of Theorem 3.1 (respectively, Theorem 3.1*).

This assertion, as well as those of Theorems 4.1 and 4.2, follows from arguments developed in the references cited above.

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